e content for students of patliputra university

B. Sc. (Honrs) Part 1 paper 2

Subject:Mathematics

Title/Heading:Groups:L'Hospital's Rule

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L'Hospital's Rule

L` Hospital rule for forms of type 0/0

Theorem Suppose that $\lim_{x\to u} f(x) = \lim_{x\to u} g(x) = 0$. If $\lim_{x\to u} \frac{f'(x)}{g'(x)}$ exists in either the finite or infinite sense (that is, if this limit is a finite number or $-\infty$ or $+\infty$), then

$$\lim_{x\to u} \frac{f(x)}{g(x)} = \lim_{x\to u} \frac{f'(x)}{g'(x)}.$$

Here u may stand for any of the symbols $a, a^-, a^+, -\infty, or +\infty$.

Problem Find $\lim_{x\to 0} \frac{e^x - 1}{x}$.

Answer Here both the numerator and denominator have limit 0. Therefore limit has $0/0$ form.

$$\therefore \quad \lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1}, \text{ applying l'Hôpital's Rule}$$

$$= \frac{e^0}{1} = 1.$$

Problem Use l'Hôpital's rule to show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Answer Here limits of both the numerator and denominator is 0. Therefore $\lim_{x\to 0} \frac{\sin x}{x}$ is in the $0/0$ form. Now

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1},$$ using l'Hôpital's Rule and noting that derivative of sin x is cos x and that of x is 1.

$$= \frac{\lim_{x\to 0} \cos x}{\lim_{x\to 0} 1},$$ using quotient rule for limits

$$= \frac{1}{1} = 1.$$

Problem Find $\lim_{x\to 0} \frac{(1+x)^n - 1}{x}$.

Answer Here both the numerator and denominator have limit 0. Therefore limit has $0/0$ form.

$$\therefore \quad \lim_{x\to 0} \frac{(1+x)^n - 1}{x} = \lim_{x\to 0} \frac{n(1+x)^{n-1}}{1}, \text{ applying l'Hôpital's Rule}$$

$$=\frac{n(1+0)^{n-1}}{1}=n.$$

Successive Application of l'Hôpital's Rule

Problem Evaluate $\lim_{x\to 0}\frac{x-\sin x}{x^3}$

Answer Here the limit is in $0/0$ form.

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}\overset{L}{=}\lim_{x\to 0}\frac{1-\cos x}{3x^2}\text{, again in } 0/0 \text{ form}$$

$$\overset{L}{=}\lim_{x\to 0}\frac{\sin x}{6x}\text{, again in } 0/0 \text{ form}$$

$$\overset{L}{=}\lim_{x\to 0}\frac{\cos x}{6}\text{, now limit can be evaluated}$$

$$=\frac{1}{6}.$$

Problem

$$\text{Find } \lim_{x\to 0}\frac{1-\cos x}{x^2+3x}.$$

Answer $\lim_{x\to 0}\frac{1-\cos x}{x^2+3x}\overset{L}{=}\lim_{x\to 0}\frac{\sin x}{2x+3}\overset{L}{=}\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$ **This is wrong,** as the first application of l'Hôpital's Rule was correct; the second was not, since at that stage the limit did not have the $0/0$ form. Here is what we should have done:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2+3x}\overset{L}{=}\lim_{x\to 0}\frac{\sin x}{2x+3}=0 \quad \textbf{This is right.}$$

Problem Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x+x^2}$

Answer The given is in the $\frac{0}{0}$ form.

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2}=\lim_{x\to 0}\frac{\sin x}{1+2x} \qquad \text{Not } \frac{0}{0}$$

$$=\frac{0}{1}=0$$

If we continue to differentiate in an attempt to apply L'Hopital's rule once more, we get

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2}=\lim_{x\to 0}\frac{\sin x}{1+2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2},$$

which is wrong.

Problem Find $$\lim_{x\to\infty}\frac{\log\left(\frac{x+1}{x}\right)}{\log\left(\frac{x-1}{x}\right)}.$$

Answer Here the given limit can be written as

$$\lim_{x\to\infty}\frac{\log\left(\frac{x+1}{x}\right)}{\log\left(\frac{x-1}{x}\right)}=\lim_{x\to\infty}\frac{\log\left(1+\frac{1}{x}\right)}{\log\left(1-\frac{1}{x}\right)}$$

and the limit is in $0/0$ form.

Also, $$\lim_{x\to\infty}\frac{\log\left(\frac{x+1}{x}\right)}{\log\left(\frac{x-1}{x}\right)}=\lim_{x\to 0}\frac{\log(x+1)-\log x}{\log(x-1)-\log x}.$$

Now we are ready to apply l'Hôpital's Rule:

$\therefore$ $$\lim_{x\to\infty}\frac{\log\left(\frac{x+1}{x}\right)}{\log\left(\frac{x-1}{x}\right)}\overset{L}{=}\lim_{x\to\infty}\frac{\frac{1}{x+1}-\frac{1}{x}}{\frac{1}{x-1}-\frac{1}{x}}$$

$$=\lim_{x\to\infty}\frac{\frac{-1}{x(x+1)}}{\frac{1}{x(x-1)}}\quad\text{(by algebraic manipulations)}$$

$$=\lim_{x\to\infty}-\frac{x-1}{x+1}=\lim_{x\to\infty}\frac{1-x}{x+1}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{x}-1}{1+\frac{1}{x}}=\frac{0-1}{1+0}=-1.$$

Exercises